Chapter 1: Quadratic forms

Exercise 2. Among the following functions, determine which ones are quadratic forms. Give their polarization.

1. $q_1 : \mathbb{C} \longrightarrow \mathbb{C}$ given by
$$q_1(z) := iz^2, \quad z \in \mathbb{C}.$$
2. $q_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by
$$q_2(z,w) := zw, \quad (z,w) \in \mathbb{C}^2.$$
3. $q_3 : \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by
$$q_3(z,w) := z\bar{w}, \quad (z,w) \in \mathbb{C}^2.$$

Solution of exercise 2.

1. For $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}$, $q_1(\lambda z) = i(\lambda z)^2 = \lambda^2 iz^2 = \lambda^2 q(z)$. $\varphi_{q_1}(u,v) = \frac{1}{2}(i(u+v)^2 - iu^2 - iv^2) = iuv$ is symmetric and bilinear. Thus q_1 is a quadratic form.
2. For $\lambda \in \mathbb{C}$ and $(z,w) \in \mathbb{C}^2$, $q_2(\lambda(z,w)) = \lambda^2 zw = \lambda^2 q_2((z,w))$. For $\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix} \in \mathbb{C}^2$, $\varphi_{q_2}(\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}) = \frac{1}{2}((z_1+z_2)(w_1+w_2) - z_1w_1 - z_2w_2) = \frac{1}{2}(z_1w_2 + z_2w_1)$ is symmetric and bilinear. Thus q_2 is a quadratic form.
3. For $\lambda \in \mathbb{C}$ and $(z,w) \in \mathbb{C}^2$, $q_3(\lambda(z,w)) = |\lambda|^2 z\bar{w} = |\lambda|^2 q_3((z,w))$. q_3 is not a quadratic form.

□

Exercise 9. Let E be a finite dimensional $\mathbb{K}$-vector space, let $q : E \longrightarrow \mathbb{K}$ be a nondegenerate quadratic form and let $h \in \mathscr{L}(E)$.

1. Show that $q \circ h$ is a quadratic form on E,
2. In which case the quadratic form $q \circ h$ is non-degenerate ?

Solution of exercise 9.

1. For $\lambda \in \mathbb{K}$ and $v \in E$, $q \circ h(\lambda v) = q \circ (\lambda h(v)) = \lambda^2 q \circ h(v)$. For $u,\ v \in E$, $\varphi_{q\circ h}(u,v) = \frac{1}{2}(q \circ h(u+v) - q \circ h(u) - q \circ h(v)) = \varphi_q(h(u), h(v))$. $\varphi_{q\circ h}(-,-) = \varphi_q(h(-), h(-))$ is bilinear and symmetric.
2. (*c.f.* Proposition 1.26) By Proposition 1.31., $q \circ h$ is non-degenerate if and only if $\det\left(\mathscr{M}_{\mathscr{B}}(\varphi_{q\circ h})\right) = \det({}^t\mathscr{M}_{\mathscr{B}}(h)\mathscr{M}_{\mathscr{B}}(\varphi_q)\mathscr{M}_{\mathscr{B}}(h)) = \det(\mathscr{M}_{\mathscr{B}}(h))^2 \det(\mathscr{M}_{\mathscr{B}}(\varphi_q)) \neq 0$. This is equivalent to $h \in \mathrm{GL}(E)$.

□

Exercise 11. Let q be a quadratic form on a $\mathbb{R}$-vector space and let φ be its polarization. Assume φ is non-degenerate, but not definite. Prove that q is not of constant sign, *i.e.* there exists $u, v \in E$ with $q(u) < 0$ ans $q(v) > 0$.

Solution of exercise 11. Since $q(u) = {}^t uMu$ is non-degenerate, the eigenvalues of M are non-zero, since q is non-definite, there are eigenvectors u and v of opposite sign, say $Mu = \lambda_1 u$ and $Mu = \lambda_2 u$ with $\lambda_1 < 0$ and $\lambda_2 > 0$. So $q(u) < 0$ and $q(u) > 0$. □

Exercise 16. Let $\Phi : \mathrm{M}_2(\mathbb{R}) \times \mathrm{M}_2(\mathbb{R}) \longrightarrow \mathbb{R}$ be defined by

$$\Phi(A,B) := \frac{1}{2}(\mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(AB)), (A,B) \in \mathrm{M}_2(\mathbb{R}) \times \mathrm{M}_2(\mathbb{R}).$$

1. Show Φ is a bilinear symmetric form on $\mathrm{M}_2(\mathbb{R})$,
2. Determine the matrix of Φ in the canonical basis of $M_2(\mathbb{R})$,
3. Prove that Φ is non-degenerate,
4. Justify why we always have
$$A^2 - \mathrm{Tr}(A)A + \det(A)I_2 = 0,$$
5. Show that the quadratic form q associated with Φ is given by $q(A) = \det(A)$,
6. Prove that for any $A, B \in \mathrm{M}_2(\mathbb{R})$, we have
$$\mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(AB) = \det(A+B) - \det(A) - \det(B).$$

Solution of exercise 16.

1. Easy to check. (Using $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$)
2. Recall that $\mathscr{B} := \{v_1 = E_{11},\ v_2 = E_{12},\ v_3 = E_{21},\ v_4 = E_{22}\}$ forms a basis of $\mathrm{M}_2(\mathbb{R})$. We compute: $\Phi(v_1,v_1) = \Phi(v_1,v_2) = \Phi(v_1,v_3) = \Phi(v_2,v_2) = \Phi(v_2,v_4) = \Phi(v_3,v_3) = \Phi(v_3,v_4) = \Phi(v_4,v_4) = 0$, $\Phi(v_1,v_4) = 1/2 = -\Phi(v_2,v_3)$. We obtain the rest by symmetry.
$$\mathscr{M}_{\mathscr{B}}(\Phi) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$
3. The associated matrix has rank 4, so Φ is non-degenerate.
4. If
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then we can write

$$
\begin{aligned}
&A^2-(a+d)A+(ad-bc)I_2\\
&=\begin{pmatrix} a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix}-\begin{pmatrix} a(a+d) & b(a+d)\\ c(a+d) & d(a+d)\end{pmatrix}+(ad-bc)I_2\\
&=\begin{pmatrix} bc-ad & 0\\ 0 & bc-ad\end{pmatrix}+(ad-bc)I_2\\
&=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}
\end{aligned}
$$

5. $q(A)=\Phi(A,A)=\frac{1}{2}(\mathrm{Tr}(A)^2-\mathrm{Tr}(A^2))=\frac{1}{2}(\mathrm{Tr}(A)^2-\mathrm{Tr}(\mathrm{Tr}(A)A-\det(A)I_2))=\frac{1}{2}(2\det(A))=\det(A)$.
6. We have $q(A+B)-q(A)-q(B)=2\Phi(A,B)$, while $q(A+B)-q(A)-q(B)=\det(A+B)-\det(A)-\det(B)$, $2\Phi(A,B)=\mathrm{Tr}(A)\,\mathrm{Tr}(B)-\mathrm{Tr}(AB)$.

□